Indian Statistical Institute, Bangalore

B. Math. Second Year

First Semester - Analysis III

Final Exam Duration : 3 hours Max Marks 100 Date : Nov 07, 2016

1. Let U, V be neighborhoods of 0 in $\mathbb{R}^n$ and let $f : U \to V$ be a diffeomorphism such that $f(0) = 0$. Then show that the restriction of f to a suitable neighborhood of 0 can be expressed as a composition of finitely many permutations and primitive diffeomorphisms. [20]

2. (a) Let $f : U \to \mathbb{R}$ be continuous function such that $\int\limits_U |f(x)|\, dx < \infty$. Here U is an open set in $\mathbb{R}^n$. Let $U_1, ..., U_k$ be open subsets of U whose union is U. Then show that $\int\limits_U f(x)\, dx = \sum\limits_{\varphi \neq I \subseteq \{1,\ ...,\ k\}} (-1)^{\sharp(I)-1} \int\limits_{\bigcap\limits_{i \in I} U_i} f(x) dx.$

   (b) Let U, V be open subsets of $\mathbb{R}^n$ and let $g : V \to U$ be a diffeomorphism. Consider the class C consisting of all open subsets U_0 of U such that, for every continuous function $f : U \to \mathbb{R}$, satisfying $\int\limits_U |f(x)| dx < \propto$, the formula
$$\int\limits_{V_0} f(x)\, dx = \int\limits_{g^{-1}(V_0)} |\ det\ g'(x)|\ f(g(x))\, dx$$
   is valid for all open subsets V_0 of U_0. Prove (without using the change of variable theorem !) that this class C is closed under finite union. You may use (with or without proof!) the result of part (a). [20]

3. (a) Give an example of a C^1 - function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f'(x)$ is invertible for all x, but f is neither one - one nor onto.

   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by $f(x, y) = (y, x)$. Show that there is no neighborhood of zero on which f can be written as the composition of two primitive diffeomorphisms. [20]

4. Define the exterior derivative $d(w)$ of a differential k- form in the class C^1. Show that if w, λ are k-form and l-form in the class C^1 then $d(w \wedge \lambda) = (dw) \wedge \lambda + (-1)^k w \wedge d\lambda$. Hence deduce that if w is in the class C^2 then $ddw = 0$. [20]

5. Let $f : U \to V$ be a homeomorphism between two open subsets of $\mathbb{R}^n$. Suppose f is in the class C^k, i.e., all the kth order (and smaller order) partial derivatives $Di_1 Di_2 ... Di_k\ f$ exist and are continuous, for all $1 \leq i_1, ..., i_k \leq n$. Then show that f^{-1} is in the class C^k. [20]